Situation: Absolute Value in the Complex Plane
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1 Prompt

This is a follow-up to situation #26 which deals with absolute value. The notion of absolute value as the distance away from 0 really comes into play in the complex plane, or $\mathbb{C}$.

2 Mathematical Foci

Mathematical Focus 1: Exploring the Absolute Value of a constant in the Complex Plane

Any number in the complex plane $\mathbb{C}$ can be written as $a+bi$ where $a, b \in \mathbb{R}$. If $a = 2$ and $b = 3.5$, $2 + 3.5i$ is here:

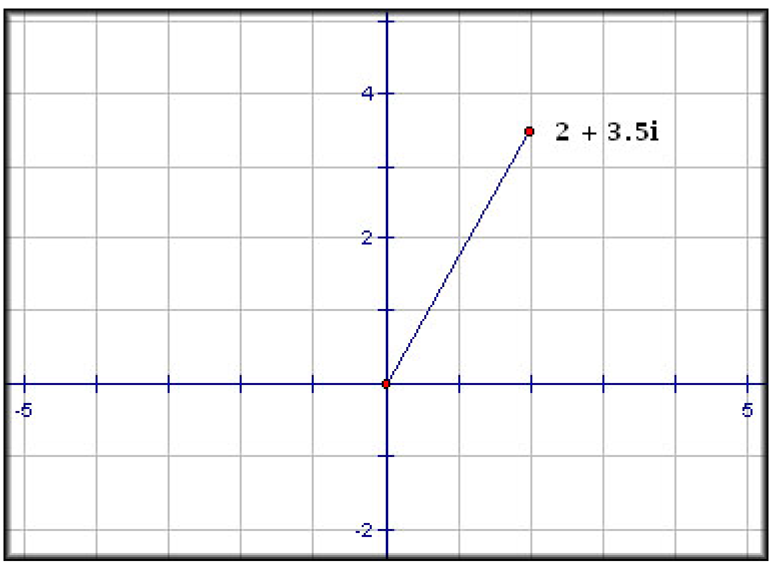

Figure 1: The position of $2 + 3.5i$

Thus, $|2 + 3.5i|$ is the distance of $2 + 3.5i$ from 0. Using the Pythagorean Theorem, we find that this distance is $\sqrt{2^2 + 3.5^2} = \sqrt{16.25}$. So,

$$|2 + 3.5i| = \sqrt{16.25}$$

Generally, given any number $a + bi \in \mathbb{C}$, $|a + bi| = \sqrt{a^2 + b^2}$.

Mathematical Focus 2: The Absolute Value Function in $\mathbb{C}$

What does $|x| = 3$ look like in $\mathbb{C}$?

This means that we are 3 units away from 0. Our graph will look be a circle with infinitely many solutions:

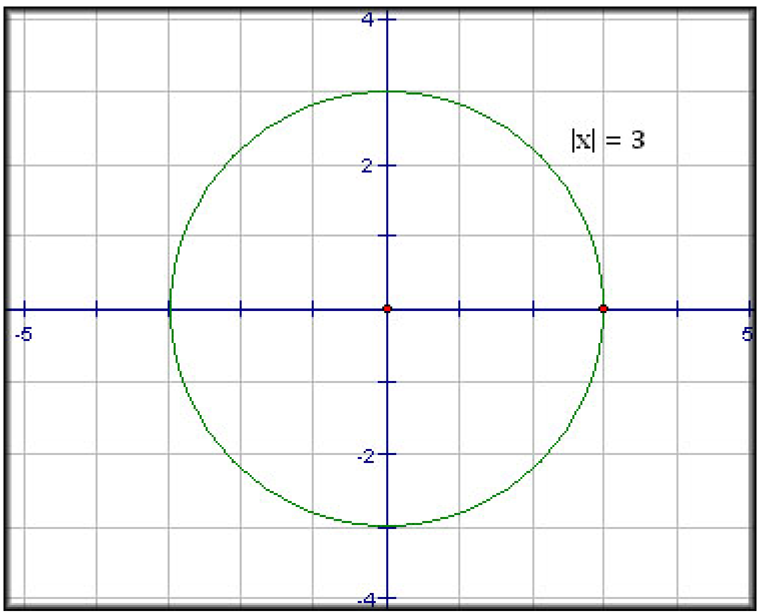

Figure 2: Graphing $|x| = 3$ in $\mathbb{C}$

Since any $x \in \mathbb{C}$ can be written in the form $x = a+bi$, and $|a+bi| = \sqrt{a^2 + b^2}$, we can solve $|x| = 3$ in the following way:

$$|x| = |a + bi| = \sqrt{a^2 + b^2} = 3$$

$$\Rightarrow$$

$$a^2 + b^2 = 9$$

Thus we have a circle of radius 3 centered at the origin.

What about the equation $|3x+1| = 5$?

Solution

Let $x = a+bi$ like before. So $|3x+1| = |3(a+bi)+1| = |3a+1+3bi|$. We regrouped our numbers so that $3a+1$ is the real part and $3bi$ is the complex part. Thus,

$$|3a+1+3bi| = \sqrt{(3a+1)^2+(3b)^2} = \sqrt{9a^2+6a+1+3b^2}$$

Now we can go back to our original equation. In steps (6) and (7), we complete the square.

$$|3x+1| = 5 \tag{1}$$
$$\sqrt{9a^2+6a+1+9b^2} = 5 \tag{2}$$
$$9a^2+6a+1+9b^2 = 25 \tag{3}$$
$$9a^2+6a+9b^2 = 24 \tag{4}$$
$$a^2+\frac{2}{3}a+b^2 = \frac{8}{3} \tag{5}$$
$$a^2+\frac{2}{3}a+\frac{1}{9}+b^2 = \frac{8}{3}+\frac{1}{9} \tag{6}$$
$$(a+\frac{1}{3})^2+b^2 = \left(\frac{5}{3}\right)^2 \tag{7}$$

This will still be a circle of radius $\frac{5}{3}$ shifted $\frac{1}{3}$ units to the left.

Mathematical Focus 3: Norms of Vectors

$|a+bi|$ can also be looked at as the *norm* of the vector $a+bi$. Students can then discuss the idea of both vectors, vector spaces, and operations on vectors. This could lead into a discussion about inner products, dot products, etc.